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Borsuk's Antipodal and Fixed-Point Theorems for Correspondences Without Convex Values

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Abstract

We present an extension of Borsuk's antipodal theorem (existence of a zero) for antipodally approachable correspondences without convex values. This result is a generalization of Borsuk-Ulam Theorem and has a fixed-point equivalent formulation.

Key words and phrases: Borsuk's antipodal Theorem, balanced set, approachable selection, fixed points.
Classification-JEL: C02, C65, C69.

The aim of this note is to extend Borsuk's Theorem to the antipodally approachable correspondences without convex values. Under suitable assumption, a correspondence with convex values is antipodally approachable. This concept is stable by composition which in not the case for correspondence with convex values. Our result generalizes those that use a correspondence with convex values.

In what follows X (resp Y) is a nonempty subset of $\mathbb{R}^n$ (resp $\mathbb{R}^p$) , capital letters $F : X \to Y$ denote correspondences while non capital letters $f : X \to Y$ will denote single-valued functions. We denote by ∂X the boundary of the subset X and $convX$ its convex hull. In the whole paper, we will assume that correspondences have nonempty values. Let $GrF = \{(x, y) \mid y \in F(x), x \in X\}$ be the graph of F, $S^n = \{x \in \mathbb{R}^{n+1} \mid \ \| x \|= 1\}$ the unit $n-$sphere and $B_X(A, r)$ the open ball of X with center A and radius r. Let $\mathcal{N}^m$ be a fundamental basis of open symmetric neighborhood of the

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origin in $\mathbb{R}^m$. A set $M \subset Y$ is said to be balanced if $\lambda M \subset M$ for every real number λ with $|\lambda| \leq 1$. Suppose that X is symmetric, a correspondence $F : X \to Y$ is said antipodal-preserving (resp. antipodal[4]) if for all $x \in X$, $F(x) = -F(-x)$ (resp. $F(x) \cap -F(-x) \neq \emptyset$). It is easy to see that if F is antipodal-preserving then F is antipodal. Note that if the correspondence F is antipodal and single-valued then it is antipodal-preserving.

We recall that a correspondence $F : X \to Y$, X and Y tow topological spaces, is upper semi-continuous (u.s.c) on X if and only if for any open subset V of Y, the set $\{x \in X : F(x) \subset V\}$ is open in X.

Definition 1 *Let X be a symmetric nonempty subset of $\mathbb{R}^n$, Y a nonempty subset of $\mathbb{R}^p$ and F a correspondence from X to Y.*

(1) For any $U \subset \mathcal{N}^n$, $V \subset \mathcal{N}^p$, a function $s : X \to Y$ is said to be a (U, V)-approximative selection of F if for any $x \in X$,
$s(x) \in (F[(x + U) \cap X] + V) \cap Y$ or equivalently $Grs \subset GrF + (U \times V)$.

(2) A correspondence $F : X \to Y$ is said to be approachable if for any $U \subset \mathcal{N}^n$, $V \subset \mathcal{N}^p$, there exists a continuous (U, V)-approximative selection for F. We denotes by $\mathcal{A}(X, Y)$ the class of such correspondences and we write $\mathcal{A}(X) = \mathcal{A}(X, X)$.

We will use the notion of approachable correspondences (see [B]).

Definition 2 *(1) A correspondence $F : X \to Y$ is said to be antipodally approachable if for any $U \subset \mathcal{N}^n$, $V \subset \mathcal{N}^p$, there exists a continuous antipodal (U, V)-approximative selection for F. We denote by $\mathcal{A}_a(X, Y)$ the class of such correspondences and we write $\mathcal{A}_a(X) = \mathcal{A}_a(X, X)$.*

(2) A correspondence $F : X \to Y$ is said to be antipodally approximable if its restriction to any symmetric compact subset K of X, $F_{|K}$, is antipodally approachable.

Remark 1 *Let Z be a symmetric subset of $\mathbb{R}^n$. Let $F : Z \to \mathbb{R}^p$ be an antipodal preserving correspondence with convex value. If there exists a continuous selection of F then there exists an antipodal-preserving selection of it. Indeed, it suffices to consider $\tilde{h}(x) = \frac{h(x)-h(-x)}{2}$ where h is the continuous selection.*

[4]It is a generalization of the original single-valued antipodal function to set-valued maps.

Proposition 1 *(1) If a correspondence $F : X \to Y$ is antipodally approachable then for any $U \subset \mathcal{N}^n$, $V \subset \mathcal{N}^p$, the correspondence $F^{U,V} : X \to Y$ defined by $(F((x+U) \cap X) + V) \cap Y$ is antipodal.*

(2) If a correspondence $F : X \to Y$ is u.s.c. with compact values and for any $U \subset \mathcal{N}^n$, $V \subset \mathcal{N}^p$, the correspondence $F^{U,V}$ is antipodal then F is antipodal.

(3) If a correspondence $F : X \to Y$ is antipodally approachable and u.s.c. with compact values then F is antipodal.

(4) Let Z be a symmetric compact subset of $\mathbb{R}^n$. If a correspondence $F : Z \to \mathbb{R}^p$ is antipodal and u.s.c. with convex value then F is antipodally approachable.

Proof:

(1) The correspondence F is antipodally approachable then for any $U \subset \mathcal{N}^n$, $V \subset \mathcal{N}^p$, there exists an antipodal-preserving function $s : X \to Y$ such that $s(x) \in (F((x+U) \cap X) + V) \cap Y$ for all $x \in X$. By the fact that $s(x) = -s(-x)$ for all $x \in X$ and the symmetry of U and V, it follows that $s(x) = -s(-x) \in \{(F((x+U) \cap X) + V) \cap Y\} \cap \{(-F((-x+U) \cap X) + V) \cap Y\}$, then $F^{U,V}$ is antipodal.

(2) Let $x_0 \in X$, for all $n \geq 1$, there exists $y_n \in \{(F((x_0 + B_X(O, \frac{1}{n})) \cap X) + B_Y(O, \frac{1}{n})) \cap Y\} \cap \{(-F((-x_0 + B_X(O, \frac{1}{n})) \cap X) + B_Y(O, \frac{1}{n})) \cap Y\}$. Therefore, there exists $t_{\frac{1}{n}} \in x_0 + B_X(O, \frac{1}{n})$ (resp. $\tilde{t}_{\frac{1}{n}} \in -x_0 + B(O, \frac{1}{n})$) and $h_n \in B_Y(O, \frac{1}{n})$ (resp. $\tilde{h}_n \in B(0, \frac{1}{n})$) such that $y_n \in F(t_n) - h_n$ (resp. $y_n \in -F(\tilde{t}_n) - \tilde{h}_n$) then $(t_n, y_n + h_n) \in GrF$ (resp. $(\tilde{t}_n, -y_n + \tilde{h}_n) \in GrF$). By a compactness argument, we can extract a subsequence $(t_{\varphi(n)}, y_{\varphi(n)} + h_{\varphi(n)})$ which converges to $(x_0, \overline{y}) \in GrF$ when $n \to +\infty$. Remark that $(\tilde{t}_{\varphi(n)}, -y_{\varphi(n)} + \tilde{h}_{\varphi(n)})$ is a subsequence of $(\tilde{t}_n, -y_n + \tilde{h}_n)$ which converges to $(-x_0, -\overline{y})$ when $n \to +\infty$. Consequently, $(-x_0, -\overline{y}) \in GrF$ and then $\overline{y} \in F(x_0) \cap -F(-x_0)$.

(3) If a correspondence $F : X \to Y$ is antipodally approachable then by (2), for any $U \subset \mathcal{N}^n$, $V \subset \mathcal{N}^p$, the correspondence $F^{U,V}$ is antipodal. Since, the correspondence F is u.s.c. with compact values then by (2), the correspondence F is antipodal.

(4) For any V in $\mathcal{N}^n$, we define as in [CH] $F^V : Z + V \to \mathbb{R}^p$ by $F^V(x) = conv(\bigcup_{z \in (x+V) \cap Z} F(z))$. Let $W_1 \in \mathcal{N}^n$ and $W_2 \in \mathcal{N}^p$, with no loss of

generality, we may assume that W_2 is nonempty convex. By lemma 1 in [CH], there exists $V \in \mathcal{N}^n$ such that $Gr\ F^V \subset Gr\ F + W_1 \times W_2$ and F^V is antipodal on Z+V with open lower sections. Let us now consider $F^V_{|Z}$ the restriction of F^V to Z, then $F^V_{|Z}$ is antipodal with convex values and open lower sections. Let us now consider the correspondence $G : Z \to \mathbb{R}^p$ defined by $G(x) = F^V_{|Z}(x) \cap -F^V_{|Z}(-x)$. This correspondence has nonempty convex values and open lower sections, hence with the Theorem of Michael [M], it has a continuous selction. Since the correspondence G is antipodal preserving then, in view of Remark 1, G (hence $F^V_{|Z}$) has a continuous antipodal selection $f : Z \to \mathbb{R}^p$. Consequently, $Gr\ f \subset Gr\ F + W_1 \times W_2$. ∎

Remark 2 *(1) Under the assumptions of Proposition 2.5 in [B], the composition of two antipodally approachable correspondences is antipodally approachable.*

(2) Remark that the class of correspondences with convex values is not stable by composition.

We will give two examples in order to show that the u.s.c. assumption (respectively compactness) can't be dropped in the assertion (2) of Proposition 1. Let $\mathcal{C}(O, r))$ be the circle in $\mathbb{R}^2$ with center $O = (0, 0)$ and radius r.

Example 1 *Define $F : \mathcal{C}(O, 1) \to \mathbb{R}^2$ by $F(x) = x$ if $x \neq (1, 0)$ and $F(1, 0) = (-1, 0)$. The correspondence F which can be viewed as a function has compact values but is not u.s.c. It is clear that for any $U \in \mathcal{N}^2$, $V \in \mathcal{N}^2$, the correspondence $F^{U,V}$ is antipodal (not antipodal-presrving) but F is not antipodal in $(1, 0)$.*

Example 2 *Define $F : \mathcal{C}(O, 1) \to \mathbb{R}^2$ by $F(x) = x$ if $x \neq (1, 0)$ and $F(1, 0) = \mathcal{C}(O, 1) \setminus (1, 0)$. The correspondence F is u.s.c with no compact values. In $(1, 0)$, it is clear that F is not antipodal but for any $U \in \mathcal{N}^2$, $V \in \mathcal{N}^2$, the correspondence $F^{U,V}$ is antipodal (not antipodal-preserving).*

We recall Borsuk's antipodal theorem:

Theorem 1 Borsuk's antipodal theorem *A single-valued antipodal continuous map $f : S^n \to \mathbb{R}^n$ has a zero value.*

The following remark will be used to extend the domain in Borsuk's antipodal theorem from S^n to the boundary of any open bounded symmetric balanced subset of $\mathbb{R}^{n+1}$.

Remark 3 *Let K be a symmetric compact subset of $\mathbb{R}^{n+1}$ and $T : S^n \to K$ a u.s.c. antipodal correspondence with convex values. If we consider T as a correspondence from S^n to $\mathbb{R}^{n+1}$ then, by Proposition 1 assertion (4), the correspondence T is antipodally approachable that is for all $\delta > 0$, $\eta > 0$, there exists a continuous $B_{\mathbb{R}^{n+1}}(O,\delta) \times B_{\mathbb{R}^{n+1}}(O,\eta)$-antipodal-preserving selection $\tilde{\varphi} : S^n \to K + B_{\mathbb{R}^{n+1}}(O,\eta)$ of T).*

Theorem 2 *Let U be an open bounded symmetric balanced subset of $\mathbb{R}^{n+1}$, then any antipodal single valued continuous function $s : \partial U \to \mathbb{R}^n$ has a zero value.*

Proof: By contradiction, suppose that for all $x \in \partial U$, $s(x) \neq 0$. Let us consider the correspondence $S : \mathbb{R}^{n+1} \to \mathbb{R}^n$ defined by $S(x) = s(x)$ if $x \in \partial U$ and $S(x) = \mathbb{R}^n$ if not. By the Dugundji extension Theorem[5] (see [DG] p. 163), there exists a continuous function that extends s over $\mathbb{R}^{n+1}$, then there exists a continuous selection of S. Since the correspondence S is antipodal preserving with convex values then by Remark 1, there exists an antipodal selection $\tilde{s}$ of S such that $\tilde{s}(x) = S(x) = s(x)$ for all $x \in \partial U$. Let $Z = \{x \in \mathbb{R}^{n+1},\ \tilde{s}(x) \neq 0\}$, then Z is an open set containing ∂U. Since ∂U is compact, then there exists $\eta > 0$ such that $\partial U + B_{\mathbb{R}^{n+1}}(0,\eta) \subset Z$.

Let $\varphi : S^n \to \partial U$ defined by $\varphi(x) = \partial U \cap \mathbb{R}_+ x$. It is easy to show that the correspondence φ is antipodal-preserving and u.s.c. with nonempty (compact) values. We will prove that φ has convex values. Let a and b in $\partial U \cap \mathbb{R}_+ x$ with $a \neq B$. Without loss of generality, we may assume that there exists α in $]0,1[$ such that $a = \alpha b$. For any c in $conv(a,b)$, there exists $\beta \in [\alpha, 1]$ such that $c = \beta b$, which leads to the existence of λ in $]0,1]$ such that $a = \lambda c$. Let us first remark that $c \in \overline{U}$ and $c \in \mathbb{R}_+ x$, since $\overline{U}$ is a balanced set and $b \in \overline{U}$. Moreover if $c \in U$ then $a \in U$ which is absurd. Consequently $c \in \partial U$ and then φ has convex values. By Remark 3, there exists a continuous antipodal selection $\tilde{\varphi} : S^n \to \partial U + B_{\mathbb{R}^{n+1}}(0,\eta)$ of φ. Finally, for all $x \in S^n$, $\tilde{s}(\tilde{\varphi}(x))$ is a continuous antipodal function without zero. This is a contradiction to Borsuk's antipodal Theorem. ∎

The main result of this paper is an extension of Theorem 2 to a correspondence. This extension generalizes Borsuk antipodal, Borsuk-Ulam Theorem [6] and Theorem 4 in [CH].

Theorem 3 *Let U be an open bounded symmetric balanced subset of $\mathbb{R}^{n+1}$ and let $F : \partial U \to \mathbb{R}^n$ be u.s.c antipodally approachable correspondence with nonempty closed values. Then F has a zero on ∂U.*

[5] Let X be any metrizable space and $A \subset X$ a closed subset. Then any continuous function $f : A \to \mathbb{R}$ has an extension $F : X \to \mathbb{R}$.

[6] see [DG] for the statement of Borsuk-Ulam Theorem and the equivalence with Borsuk antipodal Theorem.

Proof: Let $\mathbf{0}$ denote the zero function from ∂U into $\{0\}$. Suppose that F does not have zero, then $d(Gr(\mathbf{0}), Gr(F)) > 0$. Let $\varepsilon = \frac{1}{3}d(Gr(\mathbf{0}), Gr(F))$ then

$$Gr(\mathbf{0}) \cap [Gr(F) + B_{\mathbb{R}^{n+1}}(O, \varepsilon) \times B_{\mathbb{R}^n}(O, \varepsilon)] = \emptyset.$$

The correspondence F is antipodally approachable then for $V = B_{\mathbb{R}^{n+1}}(O, \varepsilon)$ and $W = B_{\mathbb{R}^n}(O, \varepsilon)$, there exists $f^{V,W}$ such that $Gr(f^{V,W}) \subset [Gr(F) + V \times W]$ which imply that $Gr(f^{V,W}) \cap Gr(\mathbf{0}) = \emptyset$. Consequently, $f^{V,W}$ is a continuous antipodal function with zero free, this is a contradiction to Theorem 2. ■

We will give two examples of u.s.c. antipodal-preserving correspondence $F : \partial B_{\mathbb{R}^3} \rightarrow \mathbb{R}^2$ with nonempty closed values. We assume that $F(x, y, z)$ depend only on z. The following two examples coincide when $z \neq 0$ but in the second example, the correspondence is antipodally approachable which is not the case in the first example.

Let us consider the following spiral ζ:

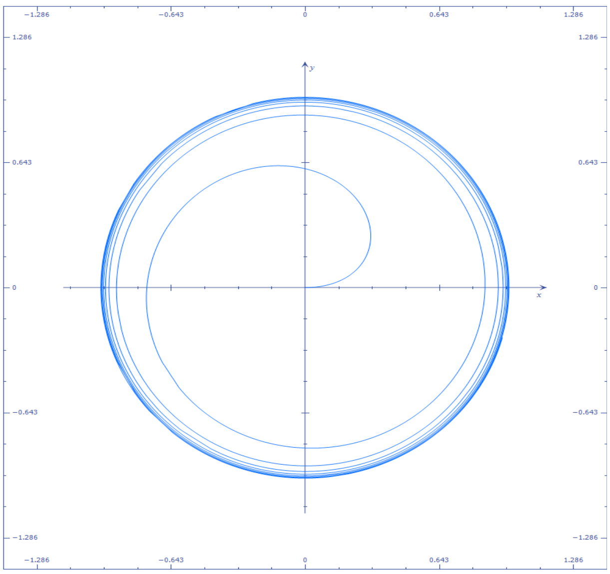
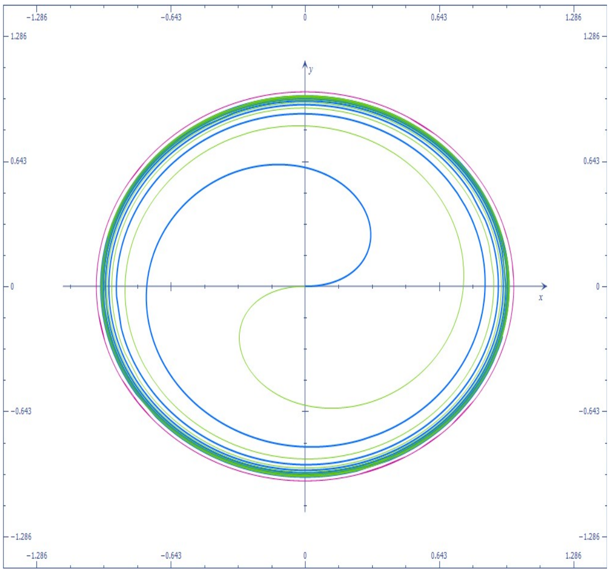


Figure 1: ζ spiral with polar equation $r = \frac{\theta}{1+\theta}$, $\theta \geq 0$.

Example 3 *We define the correspondence $F : \partial B_{\mathbb{R}^3}(O,1) \to \mathbb{R}^2$ by:*

$$F(x,y,z) = \varphi(z) = \begin{cases} \mathcal{C}(O, \sqrt{1-z^2}) \cap \zeta & \text{if} \quad z > 0 \\ \mathcal{C}(O, \sqrt{1-z^2}) \cap -\zeta & \text{if} \quad z < 0 \\ \mathcal{C}(O,1) & \text{if} \quad z = 0 \end{cases}$$



We deduce that the correspondence F is antipodal-preserving, u.s.c. with closed nonempty non convex values ($\varphi(0)$ is not convex). Note that it is easy to check that φ is a continuous antipodal-preserving function on $[-1,0[\cup]0,1]$ and that $\varphi(0)$ is the limit superior in the sense of Painlevé Kuratouski of the family $(C_n)_{n>0}$ where $C_n = \varphi(]0,\frac{1}{n}[)$[7]. Let us now prove that F is not antipodally approachable. Suppose by contradiction that the correspondence F is antipodally approachable then for all $\varepsilon > 0$, there exists a continuous antipodal function $f^\varepsilon : \partial B_{\mathbb{R}^3}(O,1) \to \mathbb{R}^2$ such that $Gr\ f^\varepsilon \subset Gr\ F + B_{\mathbb{R}^2}(O,\varepsilon) \times B_{\mathbb{R}^2}(O,\varepsilon)$. Let us now fix $\varepsilon \in]0,\frac{1}{2}[$, for each fixed $z \in [-1,1]$, we define the closed path $\gamma_z : [0,2\pi] \to \mathbb{R}^2$ by

$$\gamma_z(t) = f^\varepsilon(cos(t)\sqrt{1-z^2}, sin(t)\sqrt{1-z^2}, z),$$

[7]The limit superior of a sequence of set $(C_n)_{n>0}$ in the sense of Painlevé Kuratouski [AF] is defined by:

$$\limsup_{n\to\infty} C_n := \{x \in \mathbb{R}^2 \mid \liminf_{n\to\infty} d(x, C_n) = 0\} \ \ where \ \ d(x,X) = \inf_{y\in X} d(x,y).$$

we denote by $\tilde{\gamma}_z = \{\gamma_z(t) \mid t \in [0, 2\pi]\}$. For all $0 \le z < 1 - 2\varepsilon$, the origin $O \notin \tilde{\gamma}_z$ then let $I(\gamma_z, O) = \frac{1}{2\pi i}\int_{\gamma_z} \frac{dz}{z}$ be the index of γ_z with respect to O. For a fixed $z \in [0, 1-2\varepsilon[$, $\tilde{\gamma}_z$ is a subset of $F(z) + B_{\mathbb{R}^2}(O, 2\varepsilon)$ which is an open simply connected set [8] then γ_z is homotopic to one point a_z. Recall that in Theorem 2 p. 60 [CA], if a path $\overline{\gamma}_z$ is homotopic to a path γ_z then, $I(\gamma_z, O) = I(\overline{\gamma}_z, O)$, in particular $I(\gamma_z, O) = I(a_z, O) = 0$. By the continuity of f^ε and the fact that the index $I(\gamma_z, O)$ is a constant when γ_z is continuously deformed, we deduce that $I(\gamma_0, O) = 0$. Since $\gamma_0(t + \pi) = -\gamma_0(t)$, a simple computation of index proves that $I(\gamma_0, O)$ is odd then it can't be equal to 0.

Example 4 *Let the correspondence $F : \partial B_{\mathbb{R}^3}(O, 1) \to \mathbb{R}^2$ defined by:*

$$F(x,y,z) = \psi(z) = \begin{cases} \mathcal{C}(O, \sqrt{1-z^2}) \cap \zeta & \text{if} \quad z > 0 \\ \mathcal{C}(O, \sqrt{1-z^2}) \cap -\zeta & \text{if} \quad z < 0 \\ \mathcal{C}(O, 1) \cup ([-1,1] \times \{0\}) & \text{if} \quad z = 0 \end{cases}$$

Note that $\psi(0)$ is not convex. It is clear that the correspondence F is u.s.c. with nonempty closed values. Let us prove that the correspondence F is antipodally approachable: For every $\varepsilon > 0$, $\delta > 0$, let us consider $\eta = min(\varepsilon, \delta)$ and

$$Q = \{(x,y) \in \overline{B}_{\mathbb{R}^2}(O,1) \text{ such that } d((x,y), \psi(0)) < 1 - |\psi(2\eta)|\},$$

then $\psi(\eta) \in Q$. It is easy to see that Q is a path-connected subset of $\mathbb{R}^2$ and $O \in Q$. Then there exists a path $\gamma : [0,1] \to Q$ such that $\gamma(0) = O$ and $\gamma(1) = \psi(\eta)$. We define the function $f^\eta : \partial B_{\mathbb{R}^3}(O,1) \to \mathbb{R}^2$ by:

$$f^\eta(x,y,z) = h^\eta(z) = \begin{cases} \psi(z) & \text{if} \quad |z| \ge \eta \\ \gamma(\frac{z}{\eta}) & \text{if} \quad 0 \le z \le \eta \\ -\gamma(\frac{-z}{\eta}) & \text{if} \quad -\eta \le z \le 0 \end{cases}$$

Then f^η is a continuous, antipodal-preserving function with $Gr\ f^\eta \subset Gr\ F + B_{\mathbb{R}^2}(O, \varepsilon) \times B_{\mathbb{R}^2}(O, \delta)$. Consequently the correspondence F is antipodally approachable.

We will now focus on the correspondence from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$. The following theorem is a generalization of Theorem 6 in [CH]. Indeed, the correspondence are assumed to be approachable which is more general than a correspondence with convex values (see [CE], Remark 1, Proposition 1 assumption (4) and Remark 2).

[8] An open simply connected set D is a connected set in which every closed path is homotopic to one point in D. (see [CA] p. 61)

Theorem 4 *Let U be an open bounded symmetric balanced subset of $\mathbb{R}^{n+1}$. Let $F : \overline{U} \to \mathbb{R}^{n+1}$ be an u.s.c. correspondence with nonempty closed values and approachable on $\overline{U}$ by a function which is antipodal on ∂U. Then F has a zero value and a fixed point on $\overline{U}$.*

Proof: Let $\mathbf{0}$ denote the zero map from ∂U into $\{0\}$. Suppose that F does not have zero, then $d(Gr(\mathbf{0}), Gr(F)) > 0$. Let $\varepsilon = \frac{1}{3}d(Gr(\mathbf{0}), Gr(F))$, then $Gr(\mathbf{0}) \cap \{Gr(F) + B_{\mathbb{R}^{n+1}}(O, \varepsilon) \times B_{\mathbb{R}^{n+1}}(O, \varepsilon)\} = \emptyset$.
The correspondence F is antipodally approachable then for $V = W = B_{\mathbb{R}^{n+1}}(O, \varepsilon)$, there exists a continuous (V, W)-approximative selection $f^{V,W}$ such that $Gr(f^{V,W}) \subset Gr(F) + V \times W$ and $f^{V,W}(x) = -f^{V,W}(-x)$ for all $x \in \partial U$. Consequently $Gr(f^{V,W}) \cap Gr(\mathbf{0}) = \emptyset$ then $f^{V,W}$ is a continuous antipodal function on ∂U with zero free. This is a contradiction to Theorem 6 [CH].

It is clear that the correspondence G defined by $G(x) = F(x) - x$ for all $x \in \overline{U}$ have a zero value then the correspondence F have a fixed point. ▪

In a topological vector space, it is classical to extend the usual notion of bounded subset of a normed space using the following definition (see for exemple [K]):

Definition 3 *A subset Q of a topological vector space E is said to be bounded if for each neighbourhood U of 0 there is a $\rho > 0$ with $Q \subset \rho U$.*

Theorem 4 is easily extends to any finite dimensional vector space and this is an intermediate step towards topological vector spaces:

Proposition 2 *Let U be an open bounded symmetric balanced subset in a finite dimensional vector space E. Let $F : \overline{U} \to E$ be an u.s.c. correspondence with nonempty closed values and approachable on $\overline{U}$ by a selection which is antipodal on ∂U. Then F has a zero value and a fixed point on $\overline{U}$.*

Proof: Let $\mathcal{B} = \{x_1, \ldots, x_{n+1}\}$ be a basis of E, which allows to consider Φ the usual linear homeomorphism between $\mathbb{R}^{n+1}$ and E. If we define $V = \Phi^{-1}(U)$, it is easy to show that it is bounded in the usual sense. Moreover, letting $G = \Phi^{-1} \circ F \circ \Phi$, it is routine to check that (V, G) satisfies the assumptions of Theorem 4, which leads to the conclusion. ▪

We now extend our result to infinite dimensional space. Note that in view of Proposition 1 (4), this allows to generalize Theorem 7 of [CH].

Theorem 5 *Let M be a closed bounded symmetric balanced set in a Hausdorff locally convex topological vector space E. Let $F : M \to E$ be u.s.c. correspondence with nonempty closed values such that the closure of $F(M)$ is compact. Assume that F is approximable on M by a selection which is antipodal on ∂M, then F has at least one fixed point.*

Proof: We will construct this fixed point as a limit of "approximated fixed point". Let $\mathcal{B}$ denote a closed bounded symmetric convex neighborhood base at 0 in E. Since the closure of $F(M)$ is compact, then for each V in $\mathcal{B}$, there exists a finite subset S_V of $F(M)$ such that $(y+V)\cap S_V \neq \emptyset$ for each $y \in F(M)$. Let H_{S_V} the vector space spanned by S_V. In the following of this proof we will refer to the topology of H_{S_V}. Define $F_V : M\cap H_{S_V} \to H_{S_V}$ by $F_V(x) = (F(x)+V)\cap H_{S_V}$, then the correspondence F_V is u.s.c. with nonempty compact values. Note that $M \cap H_{S_V}$ (resp $\partial_{H_{S_V}}(M\cap H_{S_V})$, the boundary of $M\cap H_{S_V}$ with respect to the topology of H_{S_V}) is a compact subset of M (resp ∂M). Since F is approximable on M by a selection which is antipodal on ∂M then F_V is approachable on $M \cap H_{S_V}$ by a selection which is antipodal on $\partial_{H_{S_V}}(M\cap H_{S_V})$. let us remark that $0 \in M\cap H_{S_V}$, consequently either $0 \in int_{H_{S_V}}(M\cap H_{S_V})$ or $0 \in \partial_{H_{S_V}}(M\cap H_{S_V})$. In the first case, we can apply Proposition 2 with $U = int_{H_{S_V}}(M\cap H_{S_V})$ and there exists $x_V \in F_V(x_V)$. In the second case, F_V is antipodally approachable on $\partial_{H_{S_V}}(M\cap H_{S_V})$, then there exists an antipodal approximative selection s_V of F_V such that $s_V(x) \in F_V(x+V)+V$, in particular $s_V(0) = 0 \in F_V(0+V)+V$.

In both case, for each $V \in \mathcal{B}$, there exists $x_V \in M$ such that $x_V \in F_V(x_V+V)+V$. Since V is symmetric, there exists $(v,w) \in V^2$ such that $y_V \in F(z_V)$ where $y_V = x_V + v$ and $z_V = x_V + w$. A standard argument based on the compactness of $\overline{F(M)}$, the upper semicontinuity of F and the closedness of its values ends the proof (see for example [BI]). ∎

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